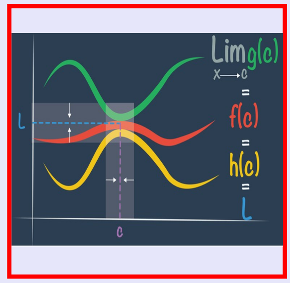
Math 261

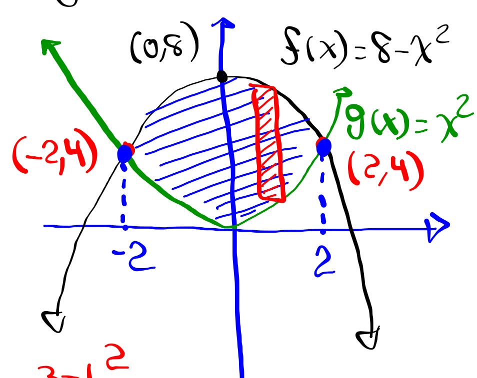
Spring 2022

Lecture 25



Class QZ 14

Find the area enclosed by $f(x) = 8 - x^2$ and $g(x) = x^2$. Drawing Required.



$$A = \int_{-2}^{2} \Big[\underbrace{8 - x^2}_{\text{Top}} - \underbrace{x^2}_{\text{Bottom}}\Big]dx$$

even

$$= 2\int_0^2 (8 - 2x^2)\,dx = 2\left[8x - \frac{2x^3}{3}\right]\Bigg|_0^2$$

$$= 2\left[8 \cdot 2 - \frac{2 \cdot 2^3}{3}\right] = 2\left[16 - \frac{16}{3}\right] = 2 \cdot \frac{32}{3} = \boxed{\frac{64}{3}}$$ ✓

Suppose $f(x)$ is a continuous function over $[a,b]$

the average value of $f(x)$ on $[a,b]$ is

$$\boxed{f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

Ex: Find f_{ave} for $f(x) = 4x - x^2$ on $[0,2]$

$f(x)$ is a polynomial function $\Rightarrow$ continuous everywhere.

$a=0, \quad b=2$

$$f_{ave} = \frac{1}{2-0}\int_0^2 (4x - x^2)\,dx$$

$$= \frac{1}{2}\left[\frac{\cancel{4}^{2}x^2}{\cancel{2}} - \frac{x^3}{3}\right]\Bigg|_0^2 = \frac{1}{2}\left[2\cdot 2^2 - \frac{2^3}{3}\right] = \frac{1}{2}\left[8 - \frac{8}{3}\right]$$

$$= \frac{1}{2}\cdot\frac{16}{3} = \boxed{\frac{8}{3}}$$

Find the average value of $f(x) = \frac{x}{\sqrt{3+x^2}}$ on $[1,3]$.

$f(x) = \frac{x}{\sqrt{3+x^2}}$

x ← continuous

$\sqrt{3+x^2}$ ← cont. everywhere

$3 + x^2 > 0$

$f(x)$ is cont. everywhere

$$f_{ave} = \frac{1}{3-1}\int_1^3 \frac{x}{\sqrt{3+x^2}}\,dx$$

Let $u = 3 + x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$x=1 \to u = 3+1^2 = 4$

$x=3 \to u = 3+3^2 = 12$

$$= \frac{1}{2}\int_1^3 \frac{x\,dx}{\sqrt{3+x^2}}$$

$$= \frac{1}{2}\int_4^{12} \frac{1}{\sqrt{u}}\,\frac{du}{2}$$

$$= \frac{1}{4}\int_4^{12} u^{-1/2}\,du$$

$$= \frac{1}{4}\cdot\frac{u^{1/2}}{1/2}\Bigg|_4^{12} = \frac{1}{2}\sqrt{u}\Bigg|_4^{12} = \frac{1}{2}\left[\sqrt{12} - \sqrt{4}\right] = \frac{2\sqrt{3} - 2}{2} = \frac{\cancel{2}(\sqrt{3}-1)}{\cancel{2}} = \boxed{\sqrt{3}-1}$$

The Mean Value Theorem for integrals:

If $f(x)$ is a continuous function on $[a,b]$, there exists a number c in (a,b) such that

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

If we multiply by $b-a$, we get

$$\int_a^b f(x)\,dx = f(c)\cdot(b-a)$$

Find c in $[0,8]$ for the function $f(x)=\sqrt[3]{x}$ such that $f(c) = f_{ave}$.

$f(x) = \sqrt[3]{x}$ is cont. everywhere

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{8-0}\int_0^8 \sqrt[3]{x}\,dx$$

$$= \frac{1}{8}\int_0^8 x^{1/3}\,dx$$

$$= \frac{1}{8}\cdot \left.\frac{x^{1/3+1}}{1/3+1}\right|_0^8$$

$$= \frac{1}{8}\cdot \left.\frac{x^{4/3}}{4/3}\right|_0^8$$

$$= \frac{3}{32}\cdot \left. x\sqrt[3]{x}\right|_0^8$$

$$= \frac{3}{32}\cdot 8\sqrt[3]{8}$$

$$= \frac{3}{32}\cdot 16 = \frac{3}{2}$$

Now $f(c) = f_{ave}$

$$\sqrt[3]{c} = \frac{3}{2}$$

$$\left(\sqrt[3]{c}\right)^3 = \left(\frac{3}{2}\right)^3$$

$$c = \frac{27}{8}$$

$\boxed{c = 3.375}$ is in $[0,8]$

Find c in $[0,2]$ such that $f(c) = f_{ave}$

for $f(x) = \dfrac{2x}{(1+x^2)^2}$.

$1+x^2>0 \Rightarrow f(x)$ cont. everywhere

$u = 1+x^2$, $du = 2x\,dx$

$x=0 \to u=1$

$x=2 \to u=5$

$$f_{ave} = \frac{1}{2-0}\int_0^2 \frac{2x}{(1+x^2)^2}\,dx$$

$$= \frac{1}{2}\cdot\int_1^5 \frac{1}{u^2}\,du$$

$$= \frac{1}{2}\int_1^5 u^{-2}\,du = \frac{1}{2}\cdot\frac{u^{-1}}{-1}\Big|_1^5 = \frac{-1}{2}\cdot\frac{1}{u}\Big|_1^5$$

$$= \frac{-1}{2}\left[\frac{1}{5}-\frac{1}{2}\right] = \frac{-1}{2}\cdot\frac{-3}{10} = \boxed{\frac{3}{5}}$$

Now $f(c) = f_{ave}$

$$\frac{2c}{(1+c^2)^2} = \frac{3}{5} \qquad 3(1+c^2)^2 = 10c$$

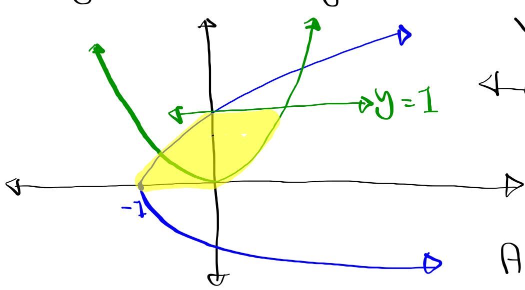
$$3(1+2c^2+c^4) = 10c$$

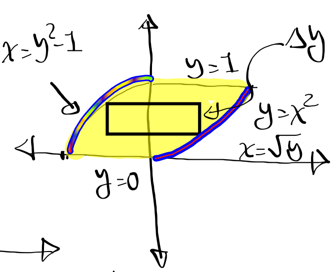
$$3c^4 + 6c^2 - 10c + 3 = 0$$

Use methods learned in Pre-Calc to find all solutions.

Make sure to finish this.

Find the area enclosed by $0 \le y \le 1$ and $x = y^2 - 1$ and $y = x^2$.





$$A = \int_0^1 (\text{Right} - \text{left})\,dy$$

$$A = \int_0^1 \left[\sqrt{y} - (y^2-1)\right]dy$$

$$= \left[\frac{y^{3/2}}{3/2} - \frac{y^3}{3} + y\right]\Big|_0^1 = \frac{1}{3/2} - \frac{1}{3} + 1$$

$$= \frac{2}{3} - \frac{1}{3} + 1 = \frac{1}{3} + 1 = \boxed{\frac{4}{3}}$$

May 11, 2022

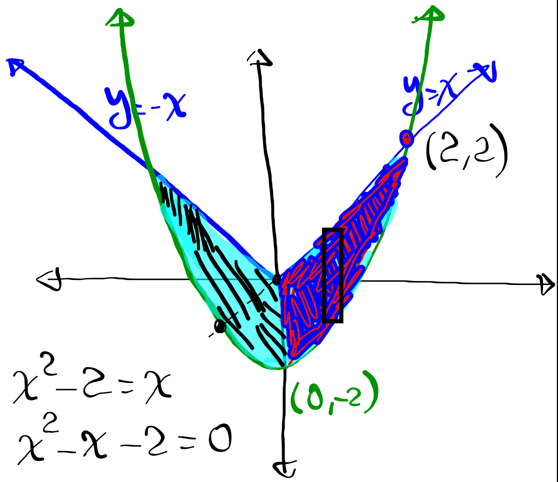
Find the area enclosed by $y=|x|$ and $y=x^2-2$.

Top Bottom

$$A=2\int_0^2 \left[x-(x^2-2)\right]dx$$

$$=2\int_0^2 (x-x^2+2)\,dx$$

$$=\boxed{}$$



$x^2-2=x$

$x^2-x-2=0$

$(x-2)(x+1)=0$

$x=2 \quad x=-1$

Evaluate $\displaystyle\int_{1/2}^{1} \frac{\cos(x^{-2})}{x^3}\,dx$

Hint:
Let $u=x^{-2}$

$du=-2x^{-3}dx$

$du=\dfrac{-2}{x^3}dx$

$\dfrac{du}{-2}=\dfrac{1}{x^3}dx$

$u=x^{-2} \Rightarrow u=\dfrac{1}{x^2}$

$x=\frac{1}{2} \Rightarrow u=\dfrac{1}{(\frac{1}{2})^2}=4$

$x=1 \Rightarrow u=\dfrac{1}{1^2}=1$

$$I=\int_4^1 \cos u \,\frac{du}{-2}$$

$$=-\frac{1}{2}\int_4^1 \cos u\,du$$

$$=-\frac{1}{2}\cdot -\int_1^4 \cos u\,du$$

$$=\frac{1}{2}\cdot \sin u\Big|_1^4=\boxed{\frac{1}{2}\left[\sin 4-\sin 1\right]}$$

Suppose $f(x)$ is continuous and $\boxed{\int_0^4 f(x)\,dx = 10}$, find $\int_0^2 f(2x)\,dx$.

$u = 2x \qquad du = 2\,dx$

$\frac{du}{2} = dx$

$\int_0^2 f(2x)\,dx$

$x = 0 \to u = 2(0) = 0$

$x = 2 \to u = 2(2) = 4$

$= \int_0^4 f(u) \cdot \frac{du}{2}$

$= \frac{1}{2} \boxed{\int_0^4 f(u)\,du} = \frac{1}{2} \cdot \boxed{10} = \boxed{5}$

If $f(x)$ is continuous and $\int_0^9 f(x)\,dx = 4$, find

1) $\int_9^0 f(x)\,dx = \boxed{-4}$

2) $\int_0^0 f(x)\,dx = \boxed{0}$

3) $\int_9^9 f(x)\,dx = \boxed{0}$

4) $\int_0^3 x f(x^2)\,dx$

$u = x^2$

$du = 2x\,dx$

$\int_0^3 x f(x^2)\,dx = \frac{1}{2}\int_0^3 2x f(x^2)\,dx$

$x = 0 \to u = 0$

$x = 3 \to u = 9$

$= \frac{1}{2}\int_0^9 f(u)\,du$

$= \frac{1}{2} \cdot 4 = \boxed{2}$

Extended Version of the Fundamental theorem of Calculus Part I:

Suppose f is continuous on $[u(x), v(x)]$ for $a \le x \le b$, and $u(x)$ & $v(x)$ are also cont for $a \le x \le b$.

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f(v(x)) \cdot v'(x) - f(u(x)) \cdot u'(x)$$

find $\displaystyle \frac{d}{dx}\int_{2x}^{3x} \frac{t}{1+t^2}\,dt$

$v(x)$

$f(t)$

$u(x)$

$$= f(3x) \cdot \frac{d}{dx}[3x] - f(2x) \cdot \frac{d}{dx}[2x]$$

$$= \frac{3x}{1+(3x)^2} \cdot 3 - \frac{2x}{1+(2x)^2} \cdot 2$$

$$= \frac{9x}{1+9x^2} - \frac{4x}{1+4x^2}$$

Find $h'(x)$ if $h(x) = \int_{\sqrt{x}}^{x^3} \cos(t^2)\,dt$

$v(x) = x^3$

$f(t) = \cos(t^2)$

$u(x) = \sqrt{x}$

$$h'(x) = f(v(x)) \cdot v'(x) - f(u(x)) \cdot u'(x)$$

$$= \cos(x^6) \cdot 3x^2 - \cos(x) \cdot \frac{1}{2\sqrt{x}}$$

$$= 3x^2 \cos x^6 - \frac{\cos x}{2\sqrt{x}}$$

Discuss the Concavity of $h(x) = \int_0^x \frac{t^2}{t^2+t+2}\,dt.$

$v(x)$

$f(t)$

$u(x)$

$$h'(x) = f(v(x)) \cdot v'(x) - f(u(x)) \cdot u'(x)$$

$$h'(x) = \frac{x^2}{x^2+x+2} \cdot 1 - \cancel{f(0) \cdot 0}^{\,0}$$

$$h'(x) = \frac{x^2}{x^2+x+2} \qquad h''(x) = \frac{2x(x^2+x+2) - x^2(2x+1)}{(x^2+x+2)^2}$$

$$h''(x) = \frac{x^2 + 4x}{(x^2+x+2)^2}$$

$h''(x) = 0$ or undefined ✓

$x^2 + 4x = 0$

$x = 0 \quad x = -4$

x		-4		0	
$h''(x)$	$+$		$-$		$+$
	C.U.		C.D		C.U.

$$h(x) = \int_{\underbrace{0}_{u(x)}}^{\overbrace{x}^{v(x)}} \underbrace{(1-t^2)\cos^2 t}_{f(t)}\, dt$$

on what interval $h(x)$ is increasing?

we need $h'(x)$

$$h'(x) = f(v(x))\cdot v'(x) - f(u(x))\cdot u'(x)$$

$$h'(x) = (1-x^2)\cos^2 x - f(0)\cdot 0$$

$$h'(x) = (1-x^2)(\cos^2 x)$$ Always non-negative

$h'(x)=0 \rightarrow 1-x^2=0$ $\quad x=\pm 1$

$\cos^2 x = 0$ $\quad x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots, \frac{(2n-1)\pi}{2}$

$n=1,2,3,\ldots$

x		-1		1	
$h'(x)$	$-$		$+$		$-$
$h(x)$	Dec.	Min	Inc	Max	Dec

$(-1, 1)$

$$f(x) = \int_0^{\sin x} \sqrt{1+t^2}\, dt,$$

$$g(y) = \int_3^y f(x)\,dx$$

Find $g''\left(\frac{\pi}{6}\right)$

$$g'(y) = f(y)\cdot 1 - f(3)\cdot 0$$

$$g'(y) = f(y)$$

$$g'(y) = \int_0^{\sin y} \sqrt{1+t^2}\, dt$$

$$g''(y) = \sqrt{1+\sin^2 y}\cdot \cos y - 0$$

$$g''(y) = \cos y \sqrt{1+\sin^2 y}$$

$$g''\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{6} \cdot \sqrt{1+\sin^2\frac{\pi}{6}}$$

$$\boxed{g''\left(\frac{\pi}{6}\right) = }$$

Class QZ 15

Evaluate $\int_0^a x\sqrt{a^2-x^2}\,dx = \int_{a^2}^{0} \sqrt{u}\cdot\frac{du}{-2}$

$u = a^2 - x^2$

$du = -2x\,dx$

$\frac{du}{-2} = x\,dx$

$x=0 \rightarrow u=a^2$

$x=a \rightarrow u=0$

$= \frac{-1}{2} \cdot - \int_0^{a^2} u^{1/2}\,du$

$= \frac{1}{2}\cdot \frac{u^{3/2}}{3/2}\Big|_0^{a^2}$

$= \boxed{\frac{1}{3}a^3}$